THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Quiz 2
Date: 6 Apr, 2017

- Time allowed: 60 minutes
- Total points: 20 points

Recall the axioms of incidence, axioms of betweenness, axioms of congruence for line segments and angles:

I1. For any distinct points A, B, there exists a unique line l_{AB} containing A, B.

I2. Every line contains at least two points.

I3. There exist three noncollinear points.

B1. If a point B is between two points A and C (written as $A * B * C$), then A, B and C are distinct points on a line, and also $C * B * A$.

B2. For any two distinct points A and B, there exists a point C such that $A * B * C$.

B3. Given three distinct points on a line, one and only one of them is between the other two.

B4. Let A, B and C be three noncollinear points, and let l be a line not containing any of A, B and C. If l contains a point D lying between A and B, then it must also contain either a point lying between A and C or a point lying between B and C, but not both.

C1. Given a line segment AB, and given a ray r originating at a point C, there exists a unique point D on the ray r such that $AB \cong CD$.

C2. If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Every line segment is congruent to itself.

C3. Given three points A, B, C on a line satisfying $A * B * C$, and three further points D, E, F on a line satisfying $D * E * F$, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

C4. Given an angle $\angle BAC$, and given a ray r_{DF}, there exists a unique ray r_{DE}, on a given side of the line l_{DF}, such that $\angle BAC \cong \angle EDF$.

C5. For any three angles α, β, γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

C6. Given triangles $\triangle ABC$ and $\triangle DEF$, suppose that $AB \cong DE$, $AC \cong DF$ and $\angle BAC \cong \angle EDF$, then $BC \cong EF$, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.

A Hilbert plane is a given set (of points) together with certain subsets called lines, and undefined notions of betweenness, congruence for line segments, and congruence for angles that satisfying all the above axioms.

1. On a Hilbert plane:

   (a) (i) Given two line segments AB and CD. State the definition of $AB > CD$.

   (ii) State the definition of an interior point of an angle $\angle BAC$.

   (iii) Given two distinct points O and A. State the definition of the circle Γ with center O and radius OA and state the definition of an interior point of Γ.

   (b) Given a line segment AB and a point O. Prove that a circle with center O and radius congruent to AB can be constructed.

   (c) (Bouns) Suppose that the Hilbert plane also satisfies the circle-circle intersection property:

   E. Given two circles Γ and γ, if Γ contains at least one interior point of γ and at least one exterior point of γ, then Γ and γ will intersect.

   Given two line segments BC and DE with $BC < DE$. Prove that an isosceles triangle $\triangle ABC$ can be constructed such that AB and AC are congruent to DE.

(10 (+3) points)

2. Given four distinct points z_1, z_2, z_3 and z_4 on $\mathbb{C}$, recall that the 4-point ratio of them $[z_1, z_2, z_3, z_4]$ is defined by $\left(\dfrac{z_4 - z_2}{z_1 - z_2}\right) / \left(\dfrac{z_4 - z_3}{z_1 - z_3}\right)$.

   (a) Show that the function $f(z) = \dfrac{1}{z}$ preserves 4-point ratio.

   (b) If $[z_1, z_2, z_3, z_4] = \lambda$, express $[z_2, z_1, z_4, z_3]$ in terms of λ.

   Hence, show that $[z_1, z_2, z_3, z_4]$ is real if and only if $[z_2, z_1, z_4, z_3]$ is real.

(6 points)

3. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathrm{Aut}(\mathbb{D}) = \{f(z) = \lambda \dfrac{z - a}{\bar{a}z - 1} \ : \ a, \lambda \in \mathbb{C}, |a| < 1, |\lambda| = 1\}$.
   Given $f(z) \in \mathrm{Aut}(\mathbb{D})$, show that if $|z| = 1$, then $|f(z)| = 1$.

(4 points)